Math 206B Lecture 4 Notes

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1 Characters of S_n

1.1 Induced M^μ representations and μ-flags

Last time, we found a character of S_4, but we didn't quite know what representation it corresponded to. Let's try to understand this a little better.

Definition 1.1. Let $\mu = (\mu_1, \dots, \mu_k)$ be a partition of n. Define

$$M^\mu = \operatorname{ind}_{S_{\mu_1} \times \cdots \times S_{\mu_k}}^{S_n} 1.$$

Example 1.1. Let $\mu = (n-1, 1)$. then $M^{n-1,1} = \operatorname{ind}_{S_{n-1}}^{S_n} 1$. $\dim(M^{n-1,1}) = n$. This is the natural representation.

Definition 1.2. Let $\mu = (\mu_1, \dots, \mu_k)$ be a partition of n. A **μ-flag** on $[n]$ is $\varnothing \subseteq A_1 \subseteq A_2 \subseteq \cdots \subseteq [n]$ such that $|A_1| + \mu_1$, $|A_2| = \mu_1 + \mu_2$, and so on.

Example 1.2. Let $\mu = (n-k, k)$, where $1 \leq k \leq n/2$. Then μ-flags are in correspondence with $(n-k)$ subsets of $[n]$: $\varnothing \subseteq A_1 \subseteq [n]$.

Example 1.3. Let $\mu = (1^n)$. μ-flags are in correspondence with S_n, where $\sigma \mapsto A_1 \subseteq \cdots \subseteq A_n$, and $A_i = \{\sigma(1), \sigma(2), \dots \sigma(i)\}$.

1.2 Structure of the M^μ representations of S_n

Definition 1.3. Let G be a finite group, and let X be a finite set. Let $G \circlearrowright X$. Then there is a **permutation representation** $\varphi : G \to S_X$.

Equivalently, there is a permutation representation $\rho_\varphi : G \to \mathrm{GL}(V)$ over $\mathbb{C}$, where $V = \mathbb{C}\langle x \rangle$.

Proposition 1.1. *M^μ is a permutation representation of S_n on μ-flags of $[n] = \{1, \dots, n\}$.*

Example 1.4. Let $\mu = (2^2)$. Then M^μ is the permutation representation of S_n on 2-subsets of $[4]$. $\dim(M^{2^2}) = \binom{4}{2} = 6$. We claim that $M^{2^2} = S^{(4)} \oplus S^{(3,1)} \oplus S^{(2^2)}$, where these refer to the irreducible representations of S_4. Let's calculate $\chi_{M^{2^2}}$:

λ	1^4	$2\,1^2$	2^2	$3\,1$	4
$\chi_{M^{2^2}}$	6	2	2	0	0
z_λ	1	6	3	8	6

Proposition 1.2. *Let $\nu = (\nu_1, \nu_2, \dots) = 1^{m_1(\nu)} 2^{m_2(\nu)} \cdots$ be a partition of n, where m_i is the number of is in ν. Then*

$$z_\nu = \frac{n!}{(m_1! 1^{m_1})(m_2! 2^{m_2}) \cdots}.$$

Theorem 1.1. $M^\mu = \bigoplus_\lambda m_{\mu,\lambda} S^\lambda$, *where*

$$m_{\mu,\lambda} := \langle M^\mu, S^\lambda \rangle = \frac{1}{n!} \sum_\nu z_\nu \chi_{M^\mu}[\nu] \chi_{S^\lambda}[\nu].$$

Theorem 1.2. *The matrix $[m_{\mu,\lambda}]$ has nonnegative integer entries and is upper triangular with 1s on the diagonal, where $\lambda \leq \mu$ is the lexicographic order.*